MATH 312
Lecture 7:

NULL-SPACE, COLUMN-SPACE & DIMENSION-COUNTING

A matrix with m Rows and n columns is a FUNCTION from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Eg: $m=2$, $n=3$, $A = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 7 & 3 \end{bmatrix}$

a) $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x+y+2z \\ 4x+7y+3z \end{bmatrix}$

3 Inputs 2 outputs.

b) $n=2$, $m=3$, $A^T = \begin{bmatrix} 2 & 4 \\ 1 & 7 \\ 2 & 3 \end{bmatrix}$

$A^T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+4y \\ x+7y \\ 2x+3y \end{bmatrix}$

2 inputs 3 outputs.

Given $B : \mathbb{R}^n \to \mathbb{R}^m$, some $m \times n$ matrix, we define two IMPORTANT subspaces, one of $\mathbb{R}^n$ and another of $\mathbb{R}^m$.

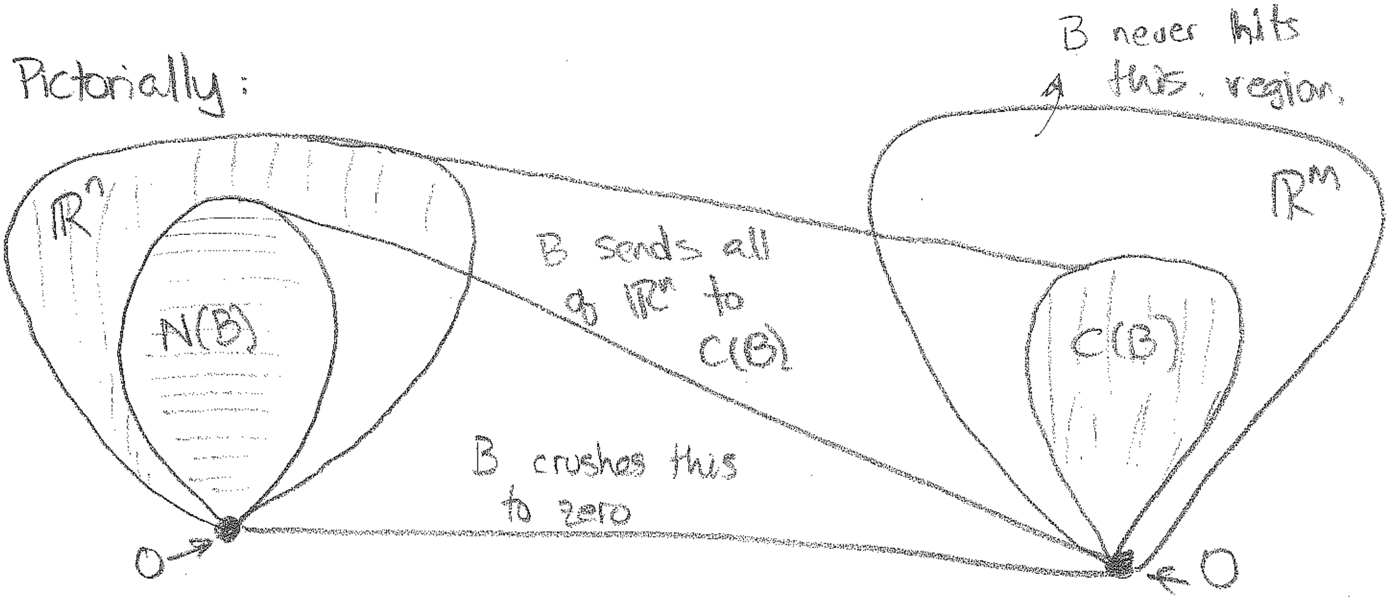
Def The NULLSPACE $N(B)$ is the collection of all vectors $\vec{v}$ in $\mathbb{R}^n$ which get sent to the zero vector in $\mathbb{R}^m$ by B. i.e,

$$\boxed{\vec{v} \text{ is in } N(B) \iff B\vec{v} = \vec{0}}$$

Def. The COLUMN SPACE $C(B)$ is the collection of all $\vec{w}$ in $\mathbb{R}^m$ that might be output by B, i.e.

$$\boxed{\vec{w} \text{ is in } C(B) \iff B\vec{x} = \vec{w} \text{ for some } \vec{x} \text{ in } \mathbb{R}^n}$$

Pictorially:



When B is "square" i.e., $m=n$ and "generic", i.e. has n pivots, then $N(B)=0$ and $C(B)=\mathbb{R}^m$. Otherwise, things get more difficult / interesting!

We can easily CHECK that N(B) is a subspace of the "domain" $\mathbb{R}^n$ while C(B) is a subspace of the "co-domain" $\mathbb{R}^m$.

- C(B) is IMPORTANT: it tells us which right sides have solutions!

Q: Why is N(B) important?

A: It lets us count "dimensions of solutions" when there are infinitely many, eg it differentiates between a "line" of solutions and a "plane" of solutions. Let's see how!

So, consider $B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 0 \\ 3 & 4 & 5 \end{bmatrix}$

- First, what is $N(B)$?

Solve $B\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so Gaussian Elimination!

$$B \longrightarrow \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$ Let's go further!

No Pivot!

We do row operations so that THE PIVOT IS THE ONLY NONZERO REPRESENTATIVE OF ITS COLUMN.

So, $R_1' = R_1 - 3R_2$ gives $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

So, to solve $B\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, we solve

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So: $$\begin{aligned} x - z &= 0 \\ y + 2z &= 0. \end{aligned}$$

VARIABLES with NO PIVOTS are called <u>FREE</u>!

Here, z is free whereas x and y are NOT.

At this stage, express the non-free variables in terms of the free ones:

$$\boxed{\begin{aligned} x &= z \\ y &= -2z \end{aligned}}$$

(once we choose z, x and y get fixed)

So, our $N(B)$ is given by ALL MULTIPLES of

$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, eg $\begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$, and ...

Let's make things "worse" by adding a row & a column to B, creating

$$D = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 2 & 1 & 0 & 5 \\ 3 & 4 & 5 & 5 \\ 1 & -2 & -5 & 5 \end{bmatrix}$$

Repeating the process from before, Gaussian elimination gives

$$D \longrightarrow \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{to make pivot the only nonzero}]{\text{extra stuff}} \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, let's say the variables are (x, y, z, w)

We see that x, y are <u>NOT</u> free but z, w are.

So,

$$\begin{aligned} x - z + 3w &= 0 \\ y + 2z - w &= 0 \end{aligned} \Rightarrow \begin{aligned} x &= z - 3w \\ y &= -2z + w. \end{aligned}$$

Hence, $N(B)$ is <u>TWO</u> - <u>DIMENSIONAL</u>, i.e,

$$\begin{bmatrix} z - 3w \\ -2z + w \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} z + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} w.$$

Returning to

$$B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 0 \\ 3 & 4 & 5 \end{bmatrix}.$$ let's solve

$$B \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}.$$ Again, we use Gaussian Elim:

on "Augmented matrix" $$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 1 \\ 2 & 1 & 0 & 4 \\ 3 & 4 & 5 & 5 \end{array}\right].$$

This produces $$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 11/5 \\ 0 & 1 & 2 & -2/5 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

To solve: $$\begin{aligned} x - z &= 11/5 \\ y + 2z &= -2/5 \end{aligned}$$

Get a PARTICULAR SOLUTION (eg $z=0$):
$$x = 11/5, \quad y = -2/5, \quad z = 0.$$

THEN the <u>GENERAL SOLUTION</u>: add in $N(B)$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11/5 \\ -2/5 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \cdot c$$

(any scalar multiple)